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Approximate second-order two-point velocity relations for turbulent dispersion

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The usual second-order two-point velocity correlations for homogeneous, isotropic turbulence in a non-divergent fluid are not applicable to tracer pairs in turbulent dispersion because on average the tracers separate as though in a divergent fluid. The present formulation accounts for the expansion that is associated with dispersion through a modification of the Karman–Howarth relations that includes the rate of expansion as an unspecified constant.

1. Introduction

The Karman–Howarth relations (Karman & Howarth 1938) concern the second-order correlations of the velocity components at the end points, 1 and 2, of a vector $\boldsymbol{r}$ (with magnitude r and components r_i) for homogeneous, isotropic turbulence in a non-divergent fluid. These relations have been used in a variety of two-particle dispersion models, as for example in those of Durbin (1980) and Kaplan & Dinar (1989). But the Karman–Howarth (K–H) relations are for a non-divergent fluid and fail to take into account the divergence of the particles that accompanies their dispersion. This divergence is associated with the Lagrangian motions of the tracers, as elaborated below, and it is not correct to relate the velocities at the two points in question by the K–H relations, which represent the statistics to be found from Eulerian measurements. To the knowledge of this author there have been no previous attempts to take this difference into account.

Karman & Howarth (1938) related the second-order correlations of the velocity components $u^*_{1,i}$ and $u^*_{2,j}$ ($i,j = 1$–3 in an arbitrary Cartesian coordinate system) to the correlation of the components parallel to $\boldsymbol{r}$, $f^* = \langle u^*_{1p} u^*_{2p}\rangle$, and perpendicular to $\boldsymbol{r}$, $g^* = \langle u^*_{1n} u^*_{2n}\rangle$. Asterisks are used to denote the non-divergent 'carrier' fluid, angle brackets are an ensemble average for a fixed length r, and subscripts p and n denote components parallel and perpendicular to $\boldsymbol{r}$, respectively, as illustrated in figure 1.

The K–H relations are

$$\overline{u^*_{1i} u^*_{2j}} = (f^* - g^*)\, r_i r_j / r^2 + g^* \delta_{ij} \tag{1}$$

and

$$g^* = f^* + \frac{r}{2}\frac{\partial f^*}{\partial r}, \tag{2}$$

where δ_{ij} is the Kroneker delta and the bar is an ensemble average for a fixed vector $\boldsymbol{r}$. By homogeneity, isotropy, and scaling u^*_{1p}, u^*_{2p}, u^*_{1n} and u^*_{2n} all have zero mean and unit variance so f^* and g^* are correlation coefficients as well as being averaged products.

The values of r are assumed to lie well within an inertial subrange of turbulence. Velocities, times, and lengths are scaled by u, T, and L, the root-mean-square

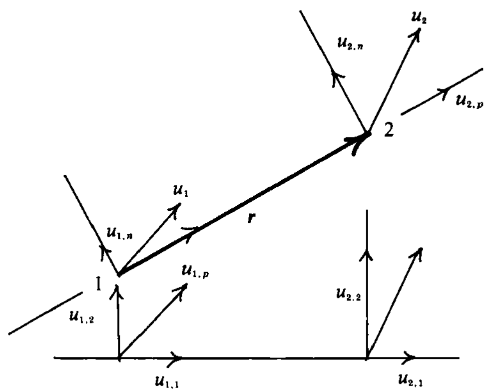

FIGURE 1. The vector $\boldsymbol{r}$ between points 1 and 2 and components of the velocities u_1 and u_2 in two dimensions.

component speed, the Lagrangian integral timescale, and the Eulerian lengthscale, respectively. It is assumed that tracer motions are everywhere the same as those of the carrier fluid, but the tracers represent a select group of fluid parcels that require special consideration because they are Lagrangian tracers.

If a cloud of tracers is injected into a turbulent carrier fluid in such a way that they initially follow the fluid motions, they will at first appear to be non-divergent. That is, for all tracer pairs having a separation r, initially $\langle u_{1p}\rangle = \langle u_{2p}\rangle = 0$ just as for the carrier fluid (asterisks). But after initial transients the tracers separate as though they were in a divergent fluid. The apparent rate of divergence is generally a function of t because the mean-square spacing of pairs of tracers is increasing and larger scales of turbulence are acting on the cloud of tracers. But for tracer pairs with a given r the average rate of separation, namely

$$\langle U\rangle = \langle \mathrm{d}r/\mathrm{d}t\rangle = \langle u_{2p} - u_{1p}\rangle > 0, \tag{3}$$

is fixed because it is determined only by r and by characteristics of the turbulence, which are assumed to be constant. The assertion in (3) that $\langle U\rangle > 0$ is an essential feature of the present model and is substantiated in Appendix A.

Two modifications of the K–H relations are required to describe the tracer correlations after the passage of any initial transients.

First, the derivation of (2), which allows us to determine $g^*(r)$ from $f^*(r)$, depends explicitly upon the assumption of a non-divergent fluid, and a modification of (2) must be derived for the expanding tracers.

Second, from the derivation of Karman & Howarth (1938) it is apparent that in a uniformly divergent fluid (1) remains formally unchanged. That is,

$$\overline{u_{1i}u_{2j}} = (f-g)\, r_i r_j/r^2 + g\delta_{ij} \tag{4}$$

applies. But average products such as $f = \langle(u_{1p}u_{2p})\rangle$ can no longer be interpreted as correlations because for an ensemble with fixed r the averaged values $\langle u_{1p}\rangle$ and $\langle u_{2p}\rangle$ are not zero. In fact, from (3) and by symmetry

$$\langle u_{2p}\rangle = -\langle u_{1p}\rangle = \tfrac{1}{2}\langle U\rangle. \tag{5}$$

It will be found necessary to express the results in terms of both the average rate of separation, $\langle U\rangle$, and $\langle U^2\rangle$, the root-mean-square expansion rate. In the absence of a complete model of dispersion these must be treated as undetermined quantities. In a particular computation model by the author, however, $\langle U \rangle$ and $\langle U^2 \rangle$ are determined by trial and error, matching input and output values, and $\langle U^2 \rangle$ is found to be proportional to $\langle U \rangle$, as discussed briefly in Appendix B.

2. The relation between $g(r)$ and $f(r)$

To modify (2) we follow the general notation of Batchelor (1953). For isotropic turbulence the second-order two-point averaged velocity-component product may be expressed as

$$\overline{u_j(\boldsymbol{x})\, u_i(\boldsymbol{x}+\boldsymbol{r})} = F(r)\, r_i\, r_j + G(r)\, \delta_{ij}, \tag{6}$$

where points 1 and 2 correspond to the positions $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{r}$, respectively. Now assume that all pairs with separation r are in a hypothetical fluid with divergence $D(r)$ as expressed by

$$\frac{\partial u_i\,(\boldsymbol{x}+\boldsymbol{r})}{\partial x_i} = \frac{\partial u_i(\boldsymbol{x}+\boldsymbol{r})}{\partial r_i} = D. \tag{7}$$

Multiplying (7) by $u_j(\boldsymbol{x})$ and averaging one obtains

$$\frac{\partial}{\partial r_i}\overline{u_j(\boldsymbol{x})\, u_i(\boldsymbol{x}+\boldsymbol{r})} = \overline{u_j(x)}\, D. \tag{8}$$

Then using (6) and the relation $\overline{u_j(\boldsymbol{x})} = \overline{u_{1p}}\, r_j/r$ one finds

$$4F + r\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial G}{\partial r} = \overline{u_{1p}}\,\frac{D}{r}. \tag{9}$$

But $\overline{u_{1p}} = \langle u_{1p} \rangle = -\tfrac{1}{2}\langle U \rangle$, and the hypothetical divergence for points with separation r and average separation speed $\langle U(r) \rangle$ is simply (see Appendix C)

$$D = 3\langle U \rangle / r, \tag{10}$$

so (9) becomes,

$$4F + r\frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial G}{\partial r} = -\frac{\tfrac{3}{2}\langle U \rangle^2}{r^2}. \tag{11}$$

Except for the non-zero right-hand side this expression is the same as that for a non-divergent fluid (Batchelor 1953).

As in the usual non-divergent development it can be shown from (6) that

$$f = r^2 F + G \quad \text{and} \quad g = G, \tag{12}$$

and by substitution into (11) one finds

$$g = f + \tfrac{1}{2} r\, \partial f/\partial r + \tfrac{3}{4}\langle U \rangle^2 \tag{13}$$

as the required revision of (2) that relates $g(r)$ to $f(r)$ for the expanding tracers.

3. The averaged cross-product $f(r)$

In this section we consider only parallel components for a fixed value of r and the subscript p is temporarily dropped.

Here we distinguish between $U^* = u_2^* - u_1^*$ for the carried fluid and $U = u_2 - u_1$ for the tracers. U and U^* are the same for any tracer pair and the carrier fluid that

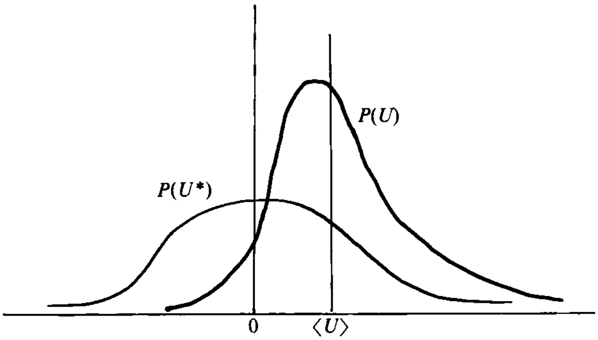

FIGURE 2. The assumed Gaussian probability density function for the separation speeds in the carrier fluid for a particular value of r with $\langle U^*\rangle = \ = 0$, and the distribution of tracer separation speeds with $\langle U\rangle > 0$. Note that the distribution of U need not be Gaussian.

contains it but they differ in their distribution functions as illustrated in figure 2. This is because the tracers represent a group of fluid parcels that are a select subset of all fluid parcels. Similarly u_2 and u_1 are subsets of u_2^* and u_1^*.

To allow us to proceed analytically we now assume that u_1^*, u_2^* and their difference U^* have Gaussian distributions. Although it is well known that u_1^* and u_2^* cannot be exactly Gaussian, the differences are slight. The more serious approximation concerns U^* which is known from dynamical theory (e.g. Monin & Yaglom 1971) to have a significant and measurable negative skewness (Anselmet *et al.* 1984). Accordingly, justification for use of the Gaussian assumption in the present context is required and this is presented in Appendix D.

The average $\langle u_1 u_2\rangle$ is derived in two steps: (a) we first find the average of u_1^* times u_2^* for a fixed value of U^*; then (b) we take an average over the distribution of U (figure 2). In (b) the asterisks disappear because the u_1 (for example) are a subset of the u_1^* over which the average is taken.

(a) Designate a fixed value of U^* as

$$U_f^* = u_2^* - u_1^*. \tag{14}$$

Here u_1^* and u_2^* are variates, but they are related by U_f^*. There are many combinations of u_1^* and u_2^* that can satisfy (14) for a specified U_f^* and we must find their averaged product. To do this represent the correlation between u_1^* and u_2^* by the formula

$$u_2^* = f^* u_1^* + q\hat{u}, \tag{15}$$

where $q = (1-f^{*2})^{\frac{1}{2}}$ and $\hat{u}$ is a hypothetical random number with the same normal distribution as u_1^* and u_2^*. One may think of (15) as saying: given values of u_1^*, the random values $\hat{u}$ will generate values of u_2^* that yield the required correlation, f^*. This can be seen by multiplying (15) by u_1^* and averaging.

Now eliminate u_2^* between (14) and (15) and solve for $\hat{u}$. Then for any selected values of U_f^* and u_1^* the required random value is given by

$$\hat{u} = (U_f^* + (1-f^*)\,u_1^*)/q. \tag{16}$$

It follows that given u_1^* the probability of obtaining U_f^* by drawing an independent random value $\hat{u}$ is

$$P(U_f^* : u_1^*) = P(\hat{u} = (U_f^* + (1-f^*)\,u_1^*)/q). \tag{17}$$

Since $\hat{u}$ is normally distributed one can write

$$P(U_f^*:u_1^*) = \frac{1}{(2\pi)^{\frac{1}{2}}}\exp\left(-\frac{(U_f^*+(1-f^*)\,u_1^*)^2}{2q^2}\right). \tag{18}$$

Now, for the fixed U_f^* an average of some function of u_1^*, say $h(u_1^*)$, is given by

$$\widetilde{h(u_1^*)} = \int_{-\infty}^{\infty}\kappa h(u_1^*)\,\mathrm{d}u_1^* \Big/ \int_{-\infty}^{\infty}\kappa\,\mathrm{d}u_1^*, \tag{19}$$

where
$$\kappa = P(u_1^*)P(U_f^*:u_1^*) = \frac{1}{2\pi}\exp\left(-\frac{u_1^{*2}}{2}-\frac{(U_f^*+(1-f^*)\,u_1^*)^2}{2q^2}\right) \tag{20}$$

and (19) can be directly integrated. Some averages of interest for fixed U_f^* are

$$\widetilde{u_2^*} = -\widetilde{u_1^*} = \tfrac{1}{2}U_f^*, \tag{21}$$

$$\widetilde{u_2^{*2}} = \widetilde{u_1^{*2}} = \tfrac{1}{2}(1+f^*)+\tfrac{1}{4}U_f^{*2} \tag{22}$$

and
$$\widetilde{u_1^*u_2^*} = \tfrac{1}{2}(1+f^*)-\tfrac{1}{4}U_f^{*2}. \tag{23}$$

(*b*) Now letting U_f^* vary, an average over the distribution of U is given by

$$\langle h(u_1^*)\rangle = \int_{-\infty}^{\infty} P(U)\,\widetilde{h(u_1^*)}\,\mathrm{d}U. \tag{24}$$

Applying (24) to (21), (22), and (23) one directly obtains (restoring the subscript p)

$$\langle u_{2p}\rangle = -\langle u_{1p}\rangle = \tfrac{1}{2}\langle U\rangle \tag{25}$$

as required from (5), and

$$\langle u_{2p}^2\rangle = \langle u_{1p}^2\rangle = \tfrac{1}{2}(1+f^*)+\tfrac{1}{4}\langle U^2\rangle, \tag{26}$$

$$f = \langle u_{1p}\,u_{2p}\rangle = \tfrac{1}{2}(1+f^*)-\tfrac{1}{4}\langle U^2\rangle. \tag{27}$$

Substituting (27) and (13) into (4) one now obtains

$$\overline{u_{1i}u_{2j}} = \left(-\frac{r}{4}\frac{\partial f^*}{\partial r}+\frac{r}{8}\frac{\partial}{\partial r}\langle U^2\rangle-\tfrac{3}{4}\langle U\rangle^2\right)\frac{r_i r_j}{r^2} + \left(\tfrac{1}{2}(1+f^*)-\tfrac{1}{4}\langle U^2\rangle+\frac{r}{4}\frac{\partial f^*}{\partial r}-\frac{r}{8}\frac{\partial\langle U^2\rangle}{\partial r}+\tfrac{3}{4}\langle U\rangle^2\right)\delta_{ij} \tag{28}$$

as the expression for the averaged cross-products of the component speeds u_{1i} and u_{2j} for dispersing tracers. The linear correlation coefficient between u_{1i} and u_{2j} also can be found from (25)–(27). The only assumption, apart from the conditions of the problem (homogeneity, etc.), is that U^*, u_1^* and u_2^* have Gaussian distributions.

4. Conclusions

One may proceed further by assuming specific formulae for $\langle U\rangle$ and $\langle U^2\rangle$ and still further by assuming the form of f^*. Appropriate formulae for the expansion are

$$\langle U\rangle = C1\,r^{\beta} \tag{29}$$

and
$$\langle U^2\rangle^{\frac{1}{2}} = C2\,r^{\beta} \tag{30}$$

where, because of the scaling of r and t, $C1$ and $C2$ are dimensionless and $r \ll 1$.

The correlation f^* is well represented in the inertial regime by

$$f^* = \exp(-r^{\alpha}) \approx 1 - r^{\alpha}.$$

It has been found (Faller & Choi 1985) that quite generally $\beta = \frac{1}{2}\alpha$, and for three-dimensional turbulence in the inertial regime $\alpha = \frac{2}{3}$. With these substitutions for $\langle U \rangle$, $\langle U^2 \rangle$, and f^*, (28) reduces to

$$\overline{u_{1i}\, u_{2j}} = r^{\frac{2}{3}}(+\tfrac{1}{6} + \tfrac{1}{12}C2^2 - \tfrac{3}{4}C1^2)\, r_i r_j/r^2 + (1 + r^{\frac{2}{3}}(-\tfrac{2}{3} - \tfrac{1}{3}C2^2 + \tfrac{3}{4}C1^2))\, \delta_{ij}. \tag{31}$$

The reader may notice that (26) and (27) do not obviously reduce to $\langle u_{2p}^2 \rangle = 1$ and $\langle u_{1p}^*\, u_{2p}^* \rangle = f^*$, respectively, for the non-divergent case $\langle U \rangle = 0$. The validity of (26) and (27) has been verified by direct numerical computation, however, by starting with u_{1p}^* and u_{2p}^* from (15) for 10^6 tracer pairs and for several values of f^*. Then a subset of data having an arbitrary distribution of U was selected in each test case to form the average $\langle U^2 \rangle$ in (26) and (27). From these calculations it was clear that when $\langle U \rangle$ was zero the $\frac{1}{4}\langle U^2 \rangle$ term always compensated the $\frac{1}{2}(1+f^*)$ term to make $\langle u_{2p}^{*2} \rangle = 1$ and $\langle u_{1p}^*\, u_{2p}^* \rangle = f^*$.

Equation (31) is suitable for use in a dispersion model with $C1$ and $C2$ regarded as constants to be determined by trial and error. Such a model, which satisfies all of the cross-products represented on the left-hand side of (31), has been proposed (Faller & Choi 1985) and some preliminary results are discussed in Appendix B. The full model, based in part upon the contents of this paper, will be presented for publication in the near future.

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Appendix A. The reality of $\langle U \rangle > 0$

Consider a cloud of tracers all with initial spacings $r_0 \ll r_f$, some fixed value. As the tracers disperse, when each pair first attains $r = r_f$ it must have $U > 0$. Once $r > r_f$ for any pair, that pair may again pass through r_f with $U < 0$, but eventually the two tracers will separate and again pass through r_f with $U > 0$. The number of pairs with $r < r_f$ must decrease with time and it is clear qualitatively that $\langle U \rangle > 0$.

The following similar thought experiment illustrates the dependence of this result upon the Lagrangian integral timescale. Consider many individual tracer pairs, each tracer moving inertially, i.e. having an infinite Lagrangian timescale. (For example one may imagine marked molecules with a mean free path much greater than r_f.) A pair with $r_0 < r_f$ and $U_0 < 0$ must change to $U > 0$ and pass through r_f with $U > 0$, as would a pair with $U_0 > 0$. All pairs with $r > r_f$ and $U_0 > 0$ do not approach r_f, but those with $U_0 < 0$ may reach r_f only to pass through r_f once again with $U > 0$. Thus in sum, $\langle U(r) \rangle > 0$. While this result is for $T \rightarrow \infty$ we should expect it to be true for any finite T, but we might expect $\langle U(r) \rangle \rightarrow 0$ for $T \rightarrow 0$. Indeed, from Taylor's (1921) well-known result

$$\overline{(x - x_0)^2} = 2u^2 tT, \tag{A 1}$$

where $\overline{(x-x_0)^2}$ is the mean-square departure of tracers from their origins at very large t, as $T \rightarrow 0$ the dispersion rate vanishes. Thus finite T and $\langle U \rangle > 0$ accompany each other and are essential ingredients of dispersion.

Appendix B. Determination of the constants $C1$ and $C2$

Preliminary calculations with the dispersion model proposed in Faller & Choi (1985) (modified to include the present formulation and other corrections) have matched input values $C1_i$ and $C2_i$ with output values $C1_0$ and $C2_0$ by trial and error. It has been found that internal consistency and the satisfaction of certain integral constraints are obtained only with the fixed ratio $C1/C2 = 1.057$. Moreover, the values of $C1$ and $C2$ are then uniquely determined by the value chosen for

$$C = L/uT,$$

a necessary input parameter for the model.

According to Corrsin (1963) the ratio L/uT is approximately 3. In view of the prominence of Corrsin in turbulence research and in the determination of the value of this ratio, it would be appropriate to refer to this ratio as the Corrsin constant. The Corrsin constant, then, is found to be the only parameter of the proposed dispersion model, as will be shown in detail in a future publication.

Appendix C. The hypothetical divergence, D

The divergence D in (10) applies only to pairs of tracers with spacing r and with appropriate Lagrangian histories. Accordingly imagine a spherical shell of diameter r containing an ensemble of many pairs of such tracers with the tracers of each pair on opposite sides of the shell. (Each pair may come from a different realization of the turbulence.) Although occasional pairs may have negative U, as discussed in Appendix A, and although all will have components not parallel to their separation vector, the majority will have $U > 0$ and the average separation speed will be $\langle U \rangle > 0$. The rate of expansion of the shell along any axis, then, will be $\langle U \rangle / r$ and the divergence of the spherical shell that is representative of tracer pairs with spacing r will be $D = 3\langle U \rangle / r$. The same average rate of separation of pairs would be found if the Eulerian velocities of similarly selected pairs were measured in a turbulent fluid having the divergence D.

Appendix D. The assumption of Gaussian distributions for u_1^*, u_2^* and U^*

Although it is well known that even isotropic turbulence cannot have exactly Gaussian velocity distributions, u_1^* and u_2^* must be symmetrical and the Gaussian assumption is thought to be not serious in this application. This assumption first enters in (15) which produces a distribution of u_2^* that is the same as that of u_1^* only if u_1^* and $\hat{u}$ are randomly selected from a Gaussian distribution. (As a counterexample, if u_1^* were selected from a box-car distribution it is clear that u_2^* could not have the same distribution except for $f^* = 1$.)

The assumption of a Gaussian U^* would seem to be more serious. The skewness of U^* in the inertial range is known to be (Monin & Yaglom 1971)

$$\langle U^{*3} \rangle = -\tfrac{4}{5}\epsilon r, \tag{D 1}$$

where the rate of dissipation, ϵ, is $O(u^3/L)$ in dimensional terms but is $O(1)$ in the present non-dimensional formulation. The normalized skewness is obtained by dividing (D 1) by

$$\langle U^{*2} \rangle^{\frac{3}{2}} = \langle (u_2^* - u_1^*)^2 \rangle^{\frac{3}{2}} = 2^{\frac{3}{2}}(1 - f^*(r))^{\frac{3}{2}}, \tag{D 2}$$

where we have used $\langle u_1^{*2}\rangle = \langle u_2^{*2}\rangle = u^2 = 1$ and $f^* = \langle u_1^* u_2^*\rangle$, and where subscript p is suppressed. Because in the inertial range

$$f^*(r) = \exp(-r^{\frac{2}{3}}) \approx 1 - r^{\frac{2}{3}} \tag{D 3}$$

it follows that the skewness is

$$S = \frac{\langle U^{*3}\rangle}{\langle U^{*2}\rangle^{\frac{3}{2}}} = -\frac{4}{5*2^{\frac{3}{2}}}O(1). \tag{D 4}$$

Thus $S = O(1)$ in the inertial range and is independent of r, as should be the case. Conversely, knowledge that S should be independent of r justifies (or can be used to derive) (D 3). We now ask whether this skewness should be a significant factor in dispersion.

Because

$$\langle U^{*3}\rangle = \langle u_2^{*3} - 3u_2^{*2}u_1^* + 3u_2^* u_1^{*2} - u_1^{*3}\rangle, \tag{D 5}$$

where $\langle u_1^{*3}\rangle$ and $\langle u_2^{*3}\rangle = 0$, it is apparent that S is determined entirely by third-order correlations. But the Gaussian approximation of U^* and the use of (15) implies that third-order correlations vanish. The issue, then, is the relative importance of third-order correlations for the proposed modifications of the K–H relations and for dispersion. For although third-order correlations are very significant dynamically, their relative importance in the kinematics of dispersion may be questioned.

The K–H relations and the dispersion model referred to in Appendix B involve only second-order correlations, and the assumption of a Gaussian U^* is therefore consistent. But we cannot prove or confidently assert that the third-order correlations are of little significance in dispersion.

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